Model Answer

MA / MSc. First Semester Examination 2014

Mathematics (AU - 6236)

Discrete Mathematical Structures - I

1 (i).

A	B	$A \to B$
T	T	T
T	F	F
F	T	T
F	F	T

(ii). A statement formula which is false regardless of the truthvalues of the statements which replace the variables in it, is called a contradiction. eg $P \wedge \neg P$.

(iii). A formula which is equivalent to a given formula and which consists of a product of elementary sums is called a conjunctive normal form of the given formula.

(iv). If α and β are strings of formulas, then $\alpha \xrightarrow{S} \beta$ is called a sequent in which α is denoted the antecent and β the consequent of the sequent.

(v). Let V be an alphabet and V^* be the set of all string over V. Let $\alpha, \beta \in V^*$. A binary operation $\circ$ on V^* is defined by $\alpha \circ \beta$, where $\alpha \circ \beta \in V^*$ is the string obtained by writing the string α on the left of the string β.

(vi). Let $(S, *)$ and $\langle T, \Delta \rangle$ be any two semigroups. A mapping $g : S \to T$ such that for any two elts $a, b \in S$

$$g(a * b) = g(a) \,\Delta\, g(b)$$

(vii). A lattice is a partially ordered set $(L, \leq)$ in which every pair of elements $a, b \in L$ has a greatest lower bound and a least upper bound.

eg. $(P(S), \subseteq)$.

(viii). A lattice is called complete if each of its non-empty subsets has a least upper bound and a greatest lower bound.

eg. every finite lattice is complete.

(ix). Let $(B, *, \oplus, ', 0, 1)$ be a Boolean algebra and $S \subseteq B$. If S contains the elements 0 and 1 and is closed under the operation $*$, $\oplus$, and $'$, then $(S, *, \oplus, ', 0, 1)$ is called a sub-Boolean algebra.

eg. For any Boolean algebra $(B, *, \oplus, ', 0, 1)$, the subset $\{0, 1\}$ and the set B are both sub-Boolean algebra.

(x) Stone's representation theorem states that any Boolean algebra is isomorphic to a power set algebra $(P(S), \cap, \cup, \sim, \phi, S)$. for some set S.

2.(a)

$\{1\}$	(1)	P	Rule P
$\{2\}$	(2)	$P \to (Q \to (R \wedge S))$	Rule P
$\{1,2\}$	(3)	$Q \to R \wedge S$	Rule T, (1), (2), $(A, A \to B \Rightarrow B)$
$\{3\}$	(4)	Q	Rule P (assumed premise)
$\{1,2,3\}$	(5)	$R \wedge S$	Rule T, (3), (4), $(A, A \to B \Rightarrow B)$
$\{1,2,3)$	(6)	S	Rule T, (5), $(A \wedge B \Rightarrow B)$
$\{1,2,3\}$	(7)	$Q \to S$	Rule CP

(b). $(P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$

$\cong \neg(P \to (Q \to R)) \vee ((P \to Q) \to (P \to R))$ $\{A \to B \cong \neg A \vee B$

$\equiv \neg(\neg P \vee (Q \to R)) \vee (\neg(P \to Q) \vee (P \to R))$

$\cong \neg(\neg P \vee (\neg Q \vee R)) \vee (\neg(\neg P \vee Q) \vee (\neg P \vee R))$

$\cong (P \wedge \neg(\neg Q \vee R)) \vee ((P \wedge \neg Q) \vee (\neg P \vee R))$

$\cong (P \wedge (Q \wedge \neg R)) \vee ((P \wedge \neg Q) \vee (\neg P \vee R))$

$\cong (P \wedge Q \wedge \neg R) \vee (P \wedge \neg Q) \vee (\neg P \vee R)$

$\cong (P \wedge ((Q \wedge \neg R) \vee \neg Q)) \vee (\neg P \vee R)$

$\cong (P \wedge ((Q \vee \neg Q) \wedge (\neg R \vee \neg Q))) \vee (\neg P \vee R)$

$\cong (P \wedge (T \wedge (\neg R \vee \neg Q))) \vee (\neg P \vee R)$ $\{T \wedge A \equiv A$

$\cong (P \wedge (\neg R \vee \neg Q)) \vee (\neg P \vee R)$

$\equiv (P \wedge \neg R) \vee (P \wedge \neg Q) \vee (\neg P \vee R)$ $\{A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$

$\equiv ((P \wedge \neg R) \vee \neg(P \wedge \neg R)) \vee (P \wedge \neg Q)$

$\cong T \vee (P \wedge \neg Q)$ $\{A \vee \neg A \cong T$

$\cong T$

3(a). Since $P \leftrightarrow Q \cong (P \to Q) \wedge (Q \to P)$
$\cong (\neg P \vee Q) \wedge (\neg Q \vee P) \cong \neg(\neg(\neg P \vee Q) \vee \neg(\neg Q \vee P))$

and $P \to Q \cong \neg P \vee Q$

and $P \wedge Q \cong \neg(\neg P \vee \neg Q)$

~~Since~~ This shows that the set of connectives $\{\neg, \wedge, \vee, \to, \leftrightarrow\}$ can be changed into the connectives $\{\neg, \vee\}$.

Similarly the set of connectives can be changed into the connectives $\{\neg, \wedge\}$.

Hence $\{\neg, \vee\}$ and $\{\neg, \wedge\}$ are complete set of connectives.

1(b).

$$P \rightarrow ((P \rightarrow Q) \wedge \neg(\neg P \vee \neg Q))$$

$$\equiv \neg P \vee ((\neg P \vee Q) \wedge (P \wedge Q)) \qquad \{A \rightarrow B \equiv \neg A \vee B$$

$$\equiv \neg P \vee (\neg P \wedge (P \wedge Q)) \vee (Q \wedge (P \wedge Q)) \qquad \text{distributive property}$$

$$\equiv \neg P \vee F \vee (P \wedge Q)$$

$$\equiv \neg P \vee (P \wedge Q)$$

$$\equiv (\neg P \wedge T) \vee (P \wedge Q)$$

$$\equiv (\neg P \wedge (Q \vee \neg Q)) \vee (P \wedge Q)$$

$$\equiv (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge Q)$$

4(a). Let $g : M \rightarrow T$ be a monoid homomorphism of $(M, *, e_m)$ onto (T, Δ, e_T) and let $z \in M$ be the zero element of M.

Therefore $z * a = a * z = z \qquad \forall a \in M$

$$\therefore \quad g(z * a) = g(a * z) = g(z)$$

$$g(z) \Delta g(a) = g(a) \Delta g(z) = g(z) \qquad \text{by monoid hom.}$$

Also any $t \in T$ can be written as $t = g(b)$ for some $b \in M$.

This shows that $g(z)$ is the required zero of T.

(b). An element 'a' in a semigroup $(S, *)$ is called an idempotent element if $a * a \equiv a^2 = a$.

Let T be the set of idempotent elements in a Comm. ~~Semigroup~~ monoid $(S, *, e)$.

~~Since $a * a = a \in T$, $\forall a \in T$~~

~~therefore the closure condition holds.~~

The associativity is an inherent condition.

Since $e * e = e$

therefore $e \in T$

Also for $a, b \in T$, we have

$$
\begin{aligned}
(a*b)*(a*b) &= a*(b*a)*b && \text{Associativity} \\
&= a*(a*b)*b && \text{Commutativity} \\
&= (a*a)*(b*b) && \text{Associativity} \\
&= a*b && \text{as } a, b \in T
\end{aligned}
$$

therefore $a*b \in T$

Hence T forms a submonoid of $(S, *, e)$.

5 (a). Let $(L, *, \oplus, 0, 1)$ be a bounded lattice.

Since $0 * 1 = 0$ and $0 \oplus 1 = 1$.

therefore 0 and 1 are complements of each other.

Let $c \in L$ be a complement of 0. then

$$0 * c = 0 \quad \text{and} \quad 0 \oplus c = 1 \quad \text{--- ①}$$

but we know that $0 \oplus c = c$ — ② as 0 is the lower bound of the lattice.

From ① & ②,

$$c = 1$$

therefore 1 is the only complement of 0.

(b). Modular inequality.

For $a, b, c \in L$ and $(L, \leq)$ be a lattice. Then

$$a \leq c \quad \text{iff} \quad a \oplus (b * c) \leq (a \oplus b) * c.$$

Proof. Let $a \leq c$ then $a \oplus c = c$ --- (1)

the distributive inequality is given by

$$a \oplus (b * c) \leq (a \oplus b) * (a \oplus c) \quad \text{--- (2)}$$

Put (1) in (2), we get

$$a \oplus (b * c) \leq (a \oplus b) * c$$

Conversely, we know

$$a \leq a \oplus (b * c) \quad \text{--- (3)}$$

— (4)

$$\text{and } (a \oplus b) * c \leq c$$

and given inequality $a \oplus (b * c) \leq (a \oplus b) * c$ — (5)

From (3), (4) & (5), we get

$$a \leq c.$$

6(a). $(a*b)' \oplus (a\oplus b)'$

$= (a'\oplus b') \oplus (a' * b')$ by De'Morgan's law

$= ((a'\oplus b') \oplus a') * ((a'\oplus b') \oplus b')$ distributive law

$= (a'\oplus b') * (a'\oplus b')$

$= a'\oplus b'$

(b). Let $(L, *, \oplus)$ be a distributive lattice, then

$$(a * b) \oplus c = (a\oplus c) * (b\oplus c) \quad \text{---}(1)$$

Also given $a*b = a*c \quad \text{---}(2)$

$a\oplus b = a\oplus c \quad \text{---}(3)$

from (1) we get

$(a*b)\oplus c = (a\oplus c) * (b\oplus c)$

$(a*c) \oplus c = (a\oplus b) * (b\oplus c)$ using (2) & (3)

$c = b\oplus (a*c)$ absorption & distributive

$c = b \oplus (a*b)$ by (2)

$c = b$

7(a). (i) $a = 0 \Leftrightarrow ab' + a'b = b$

Let $a = 0$ then $a' = 1$

therefore $ab' + a'b = 0.b' + 1.b$

$= 0 + b = b$

Conversely, let $ab' + a'b = b \quad \text{---}(1)$

$(ab' + a'b)\, b' = bb'$

$ab'b' + a'bb' = 0$ as $bb' = 0$

$ab' + 0 = 0$

$ab' = 0 \quad \text{---}(2)$

From (1)

$$(ab' + a'b)\, b = bb$$

$$ab'b + a'bb = bb$$

$$0 + a'b = b$$

$$a'.b = b \quad \text{--- (3)}$$

$$a(a'b) = ab$$

$$0 = ab \quad \text{--- (4)}$$

From (2) & (4)

$$ab' + ab = 0$$

$$a(b + b') = 0$$

$$a \cdot 1 = 0$$

$$a = 0$$

(ii). $(a + b')(b + c')(c + a') = (a' + b)(b' + c)(c' + a)$

both sides will be shown equal to $abc + a'b'c'$.

Q. Give definition of a (phrase structure) grammar. Also discuss the context sensitive, context free, and regular grammar with examples.